B.Sc. (Semester - 5)
Subject: Physics
Course: US05CPHY21
Classical Mechanics

UNIT- I LAGRANGIAN FORMULATION

Introduction:

We can write the equation of motion of a particle in terms of the Cartesian or polar coordinates. Consider the motion of a particle in a central force field. We studied this motion by using plane polar coordinates *r* and Ѳ. The motion of a projectile is considered in the Cartesian coordinate system. Hence particular coordinate system chosen to simplify the problem. Such a dependence on the coordinate system is undesirable.

In the Lagrangian formulation, we should write the equation of motion without any specific reference to the coordinate system used. This is the approach in the Lagrangian formulation of classical mechanics. There are number of advantages over the Newtonian formulation. The Lagrangian formulation is of a very general nature and makes use of generalized coordinates and velocities which are independent of the coordinate system.

Constraints:

The motion of a free particle is described by three independent coordinates*x,y,z* in Cartesian system or *r*,θ, ϕ in polar coordinates. The particle is free to execute motion along any axis. The particle has three degrees of freedom.

The number of independent ways in which a mechanical system can move without violating any constraints is called the number of ***degrees of freedom****.*

In other words, the numbers of degrees of freedom is the number of independent variables to describe the positions and velocities of all the particles.

Examples:

- For the system of N particles, the number of degrees of freedom is 3N.
- For a particle constrained to move on a plane then two variables *x,y* or *r*, θ are sufficient to describe its motion. The particle has two degree of freedom. Hence, the constraints on the motion of the particle in a plane, reduces the number of degrees of freedom by one.
- If the particle be tied to one end of a rigid rod, then particle is move along a circle. This motion is described by single variable θ and has only one degree of freedom.

When the motion of a system is restricted in some way are known as ***constraints.*** A constrained motion is a motion which cannot proceed arbitrarily in any manner. The particle motion is restricted to occur only.

Example: A bead sliding down a wire, a disc rolling down an inclined plane, the motion of simple pendulum, motion of spherical pendulum, motion of rigid body etc. are examples of constrained motion.

When constraints are introduced into a system its number of degree of freedom is reduced.

- In the case of rigid body, the constraint is that the distance between any two particles of the body is constant. It can be written as,

$$|\vec{r}_i - \vec{r}_j|^2 = |\vec{r}_{ij}|^2 = (x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 = contant \qquad \dots (1.1)$$

  Where, $\vec{r}_i$ and $\vec{r}_j$ are the position vectors of i^{th} and j^{th} particles respectively.

- A simple pendulum moving in XY-plane as shown in fig.1.1, the two equation of the constraints are,

$$z = 0 \; and \; x^2 + y^2 = l^2 = contant \qquad \dots (1.2)$$

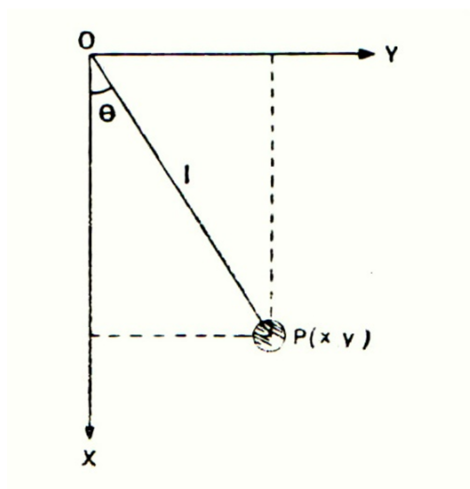

Fig. 1.1 – Constrains on a simple pendulum

Here, one variable θ is sufficient to locate oscillating particle P.

- The equation of constraints in the case of a particle moving on or outside the surface of a sphere of radius '*a*' is

$$x^2 + y^2 + z^2 \geq a^2 \qquad \dots (1.3)$$

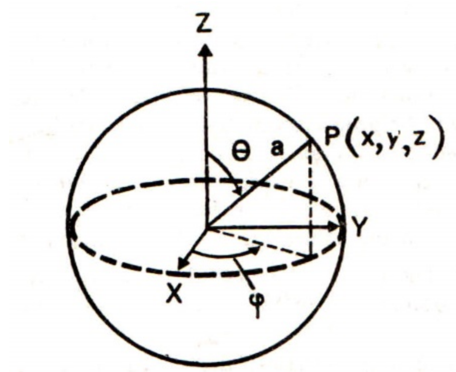

Fig. 1.2 – Generalised coordinates of particle constrained to move on the surface of a sphere

- If the particle is in contact with the surface of the sphere then,
$$r^2 = x^2 + y^2 + z^2 = a^2$$
- If the particle leaves the surface then,
$$r^2 = x^2 + y^2 + z^2 > a^2$$

Thus, a constraint is a restriction on the freedom of motion of the system in the form of a condition.

(a) Holonomic and Non-Holonomic constraints:

- A constraint which can be expressed in the form of an equation relating the coordinates of the system and time in the following way is called holonomic constraint.
  The general form of equation for a system of N particle is
  $$F_i[x_1, y_1, z_1, x_2, y_2, z_2, \dots \dots . x_N, y_N, z_N, t] = 0 \qquad \dots (1.4)$$
  where $i = 1,2,3 \dots \dots . k$ and F_i is some function of the coordinates. Here, *i* denotes the *i*th constraint.
- A constraint which cannot be expressed in the form of an equality relating the coordinates of the system and time is called holonomic constraint. It may be in the form of inequality.
- If there are k constraints, the number of degrees of freedom is reduced to (3N-K). Hence, instead of 3N coordinates, we can assign the (3N-K) independent variables like $q_1, q_2, q_3, \dots \dots \dots q_{3N-k}$ to describe the system. These variables have not the dimensions. Such variables are called the generalized coordinates.
  For examples, $q = \theta$ in simple pendulum, $q_1 = r$ and $q_2 = \theta$ in the case of motion of a particle in a central force field are generalized coordinates.
- A set of independent coordinates $q_1, q_2, q_3, \dots \dots \dots q_{3N-k}$ is called a proper set of generalized coordinates.

  The constraints cannot be always written in terms of coordinates but also in terms of velocities.
  For example, a disc of radius $'a'$ rolling down from an inclined plane as shown in fig.1.3

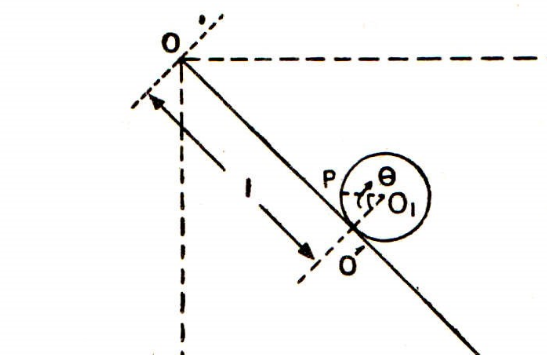

Fig. 1.3 – Generalised coordinates of a disc rolling down an Inclined plane without slipping

The equation of constraint is

$$\frac{dl}{dt} = a\dot{\theta}, \qquad \text{or} \qquad dl = a\,d\theta \qquad \ldots(1.5)$$

Integrating this equation, we get

$$l - a\theta = contant \qquad \ldots(1.6)$$

This is holonomic constraint.

(b) Scleronomous and Rheonomous constraints:

- When the constraints are independent of time are known as Scleronomous constraints.
- When the constraints are depends on the time are known as Rheonomous constraints.
- The constraints in the case of rigid body areScleronomous constraint while that of a bead of a rotating wire loop is Rheonomous.
- If we construct a simple pendulum whose length changes with time i.e. $l = l(t)$, then constraint is time dependent. If the radius of the sphere is changing with time i.e. $r = r(t)$, the constraint is also time dependent. These are Rheonomous constraints.

Generalized Coordinates:

For a system of N-particles, there are '*k*' constraints then all the 3N coordinates of all the N-particles in the system are not independent. The forces of constraints are not always known because they may depend upon the motion itself. In such a case, we can introduce a proper set of variables $q_1, q_2, \ldots \ldots \ldots q_n$ where, $n = 1,2, \ldots \ldots (3N - 1)$ are called generalized coordinates. These 3N-k variables describe the system completely.

The transformation equations can be written as

$$\left.\begin{aligned} x_i &= x_i(q_1, q_2, \ldots \ldots \ldots . q_n, t) \equiv x_i(q_j, t) \\ y_i &= y_i(q_1, q_2, \ldots \ldots \ldots . q_n, t) \equiv y_i(q_j, t) \\ z_i &= z_i(q_1, q_2, \ldots \ldots \ldots . q_n, t) \equiv z_i(q_j, t) \end{aligned}\right\} \qquad \ldots(1.7)$$

Above equation can be reduced to a single vector equation as

$$\left.\begin{aligned}\vec{r}_i = \vec{r}_i(q_1, q_2, \dots \dots \dots q_n, t)\\ \therefore\ \vec{r}_i = \vec{r}_i(q_j, t)\end{aligned}\right\} \qquad \dots (1.8)$$

Where, $i = 1,2, \dots \dots \dots N$ and $j = 1,2, \dots \dots \dots n$

Illustrations:

1. **Simple Pendulum:**

The motion of a simple pendulum oscillating in a vertical plane can be described in terms of Cartesian coordinates x and y.

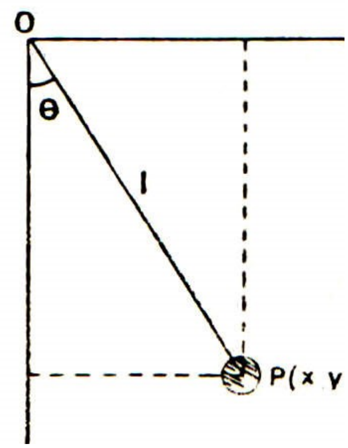

Fig. 1.4 – Generalised coordinates of particle constrained to move on the surface of a sphere

But

$$x = lcos\theta \quad \text{and} \quad y = lsin\theta \qquad \dots(1.9)$$

Since, l is a constant, the only variable involved is θ. It can be chosen as the generalized coordinate.

$$q = \theta = \cos^{-1}\frac{x}{l} \quad or \;\; q = \theta = \sin^{-1}\frac{y}{l} \qquad \dots (1.9)$$

2. **Spherical Pendulum:**

The motion of a particle constrained to move on the surface of the sphere of radius 'a' , can be described in terms of Cartesian coordinates x, y and z. The relation is

$$x^2 + y^2 + z^2 = a^2$$

Only two coordinates are independent.

Let us introduce,

$$q_1 = \frac{x}{a} \;\; and \; q_2 = \frac{y}{a}$$

$$\text{Then,} q_3 = \sqrt{\frac{a^2-x^2-y^2}{a^2}} = \sqrt{1-q_1^2-q_2^2} \qquad \dots (1.10)$$

Hence, q_3 is not an independent coordinate.

The Cartesian coordinates are not convenient in this case because of the spherical symmetry and hence spherical polar coordinates are used. Here, $r = a = constant$

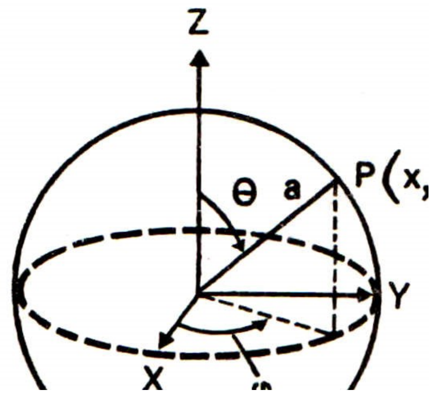

Fig. 1.5 – Constrains on a simple pendulum

The generalized coordinates will be $q_1 = \theta$ and $q_2 = \phi$

$$\text{Therefore,} \qquad \left.\begin{aligned} q_1 &= \theta = \cos^{-1}\frac{z}{a} \\ q_2 &= \phi = \sin^{-1}\frac{y}{x} \end{aligned}\right\} \qquad \dots (1.11)$$

3. **Inclined Plane:**

Consider a disc rolling down an inclined plane without slipping. We required two coordinates to specify the position of any point on the rim. The disc has translation and rotational motion. We can use both Cartesian and polar coordinates.

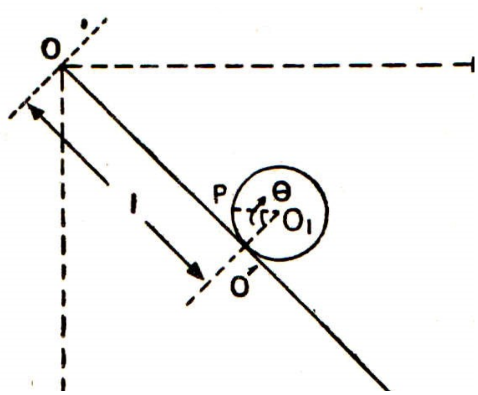

Fig. 1.6 – Generalised coordinates of a disc rolling down an Inclined plane without slipping

Any point P can be located by giving the position of the centre of disc O_1or the point of contact $O^{'}$ from reference point O and the angle made by the radial line with some fixed line.

The equation of constraints are, $z = 0$ and $l = \sqrt{x^2 + y^2}$,where l is the distance travelled by the disc down the plane.

For spherical polar coordinates $\emptyset = 0$ and $O_1P = constant$ are the equation of constraints. Thus, l and θ are the proper generalized coordinates for this system.

- The inverse transformation equations can be written as

$$q_j = q_j(x_1, y_1, z_1, x_2, y_2, z_2, \ldots \ldots x_N, y_N, z_N, t)$$

$$\therefore \quad q_j = q_j(\vec{r}_1, \vec{r}_2, \ldots \ldots \vec{r}_N, t) \qquad \ldots (1.12)$$

Where, $j = 1,2, \ldots \ldots n$

- The time derivatives of the generalized coordinates $\dot{q} = \frac{dq}{dt}$ are called the generalized velocities.

System Point and Configuration Space:

The determination of the motion of a single particle in three dimensions is a mechanical problem. The mechanical problem involving two particles, every particle being described by a set of three coordinates, can be reduced to a single particle problem simply by considering that the single particle moves in a six-dimensional space. Thus, in general, a problem involving N-particles can be treated as one of a 'Single Particle' moving along a trajectory in 3N dimensional space. This space is called 'Configuration Space' and the single particle as 'System Point' in 'Configuration Space' is called the motion of the system between any two given instants. Configuration space has no necessary connection with the three-dimensional space.

D'Alembert's Principle:

Consider a system described by n generalised coordinates $q_j\ (j = 1,2, \ldots \ldots n)$. Suppose the system undergoes a certain displacement in the configuration space in such a way that it does not take any time. Such displacement is called virtual displacements because they do not represent actual displacements of the system. Since there is no actual motion of the system, the work done by the forces of constraint in such a virtual displacement is zero.

Now, suppose the system is in equilibrium, i.e. total force on every particle is zero. Then work done by this force in a small virtual displacement $\delta\vec{r}_i$ will also vanish.

$$\therefore \quad dW = \sum_i \vec{F}_i \cdot \delta\vec{r}_i = 0 \qquad \ldots (1.13)$$

Let this total force be expressed as sum of applied force $\vec{F}_i^a$ and force of constraints $\vec{f}_i$. Then equation (1.13) becomes

$$\sum_i \vec{F}_i^a \cdot \delta\vec{r}_i + \sum_i \vec{f}_i \cdot \delta\vec{r}_i = 0 \qquad \dots(1.14)$$

Let us assume that the virtual work done by the force of constraints is zero.

$$\therefore \quad \sum_i \vec{f}_i \cdot \delta\vec{r}_i = 0 \qquad \dots(1.15)$$

Equation (1.15) will not hold if frictional forces are present. This is because the frictional forces act in a direction opposite to that of the displacement.

The principle of virtual work states that, *virtual work done by the applied forces acting on a system in equilibrium is zero, provided that no frictional forces are present.*

$$\text{Hence,} \sum_i \vec{F}_i^a \cdot \delta\vec{r}_i \ = 0 \qquad \dots(1.16)$$

The equation is termed as principle of virtual work. To interpret the equilibrium of the systems, D'Alembert adopted an idea of a reversed force. He conceived that a system will remain in equilibrium under the action of a force equal to the actual force $\vec{F}_i$plus a reversed effective force $\dot{\vec{p}}_i$. Thus,

$$\vec{F}_i + \left(-\dot{\vec{p}}_i\right) = 0$$

$$\therefore \qquad \vec{F}_i - \dot{\vec{p}}_i = 0 \qquad \dots(1.17)$$

Thus, the principle of virtual work takes the form

$$\sum_i (\vec{F}_i - \dot{\vec{p}}_i) \cdot \delta\vec{r}_i \ = 0 \qquad \dots(1.18)$$

Equation (1.18) is the mathematical statement of D'Alembert principle. In this equation all forces $\vec{F}_i$ are the applied forces. The forces of constraints do not appear in this equation.

- We have,

$$\vec{r}_i = \vec{r}_i(q_j, t)$$

The velocities are given by

$$\vec{v}_i = \dot{\vec{r}}_i = \sum_j \frac{\partial \vec{r}_i}{\partial q_j}\dot{q}_j + \frac{\partial \vec{r}_i}{\partial t}$$

Where, $\dot{q}_j = \frac{\partial q_j}{\partial t}$ are called the *generalized velocities*.

- The virtual work done by force by forces $\vec{F}_i$ in terms of virtual displacement $\delta\vec{r}_i$ is given by

$$\delta W = \sum_i \vec{F}_i \cdot \delta\vec{r}_i$$

$$= \sum_j \left(\sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \right) \delta q_j$$

$$\delta W = \sum_i Q_j \, \delta q_j$$

$$where, \; Q_j = \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}$$

Here, $Q_j$is called the *generalized force*

Lagrange's Equation:

The coordinates transformation equations are

$$\vec{r}_i = \vec{r}_i(q_1, q_2, \dots \dots \dots \, q_n, t)$$

$$\therefore \; \frac{d\vec{r}_i}{dt} = \frac{\partial \vec{r}_i}{\partial q_1}\frac{dq_1}{dt} + \frac{\partial \vec{r}_i}{\partial q_2}\frac{dq_2}{dt} + \cdots \dots \dots + \frac{\partial \vec{r}_i}{\partial t}\frac{dt}{dt}$$

$$\therefore \quad \vec{v}_i = \sum_j \frac{\partial \vec{r}_i}{\partial q_j}\dot{q}_j + \frac{\partial \vec{r}_i}{\partial t} \qquad \dots (1.19)$$

An infinitesimal displacement $\delta\vec{r}_i$ can be connected withδq_j as,

$$\delta\vec{r}_i = \sum_j \frac{\partial \vec{r}_i}{\partial q_j} \, \delta q_j + \frac{\partial \vec{r}_i}{\partial t}\delta t$$

But last term will be zero because in virtual displacement only coordinate displacement is considered and not that of time.

$$\therefore \quad \delta\vec{r}_i = \sum_j \frac{\partial \vec{r}_i}{\partial q_j} \, \delta q_j \qquad \dots (1.20)$$

Now, D'Alembert principal is

$$\sum_i (\vec{F}_i - \dot{\vec{p}}_i) \cdot \delta\vec{r}_i \; = 0 \qquad \dots (1.21)$$

Using equation(1.20) in (1.21), we get

$$\sum_i (\vec{F}_i - \dot{\vec{p}}_i) \cdot \frac{\partial \vec{r}_i}{\partial q_j} \, \delta q_j = 0$$

$$\therefore \quad \sum_{i,j} \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \, \delta q_j - \sum_{i,j} \dot{\vec{p}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \, \delta q_j = 0 \qquad \dots (1.22)$$

Now, we define the component of generalized force as,

$$\sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = Q_j \qquad \dots (1.23)$$

Using equation (1.23) in (1.22), we get

$$\sum_j Q_j \, \delta q_j - \sum_{i,j} \dot{\vec{p}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \, \delta q_j = 0 \qquad \dots (1.24)$$

Let us evaluate the second term of equation (1.24) as follows

$$\sum_{i,j} \dot{\vec{p}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \, \delta q_j = \sum_{i,j} m_i \, \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \, \delta q_j$$

$$= \sum_{i,j} \left\{ \frac{d}{dt}\left(m_i \dot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \right) - m_i \dot{\vec{r}}_i \cdot \frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j} \right) \right\} \delta q_j$$

$$= \sum_{i,j} \left\{ \frac{d}{dt}\left(m_i \vec{v}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \right) - m_i \vec{v}_i \cdot \frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j} \right) \right\} \delta q_j \qquad \dots (1.25)$$

$$But, \qquad \frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j} \right) = \frac{\partial \vec{v}_i}{\partial q_j} \qquad \dots (1.26)$$

Also differentiating equation (1.19) with respect to $\dot{q}_j$, we get

$$\frac{\partial \vec{v}_i}{\partial \dot{q}_j} = \frac{\partial \vec{r}_i}{\partial q_j} \qquad \dots . (1.27)$$

Putting equations (1.26) and (1.27) in equation (1.25), we get

$$\sum_{i,j} \dot{\vec{p}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \, \delta q_j = \sum_{i,j} \left\{ \frac{d}{dt}\left(m_i \vec{v}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \right) - m_i \vec{v}_i \cdot \frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j} \right) \right\} \delta q_j$$

$$= \sum_j \left[\frac{d}{dt}\left\{ \frac{\partial}{\partial \dot{q}_j}\left(\sum_i \frac{1}{2} m_i v_i^2 \right) \right\} - \frac{\partial}{\partial q_j}\left(\sum_i \frac{1}{2} m_i v_i^2 \right) \right] \delta q_j$$

With this substitution equation (1.24) becomes

$$\sum_j Q_j \delta q_j - \sum_j \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j}\right] \delta q_j = 0 \qquad \ldots (1.28)$$

$$Here, \sum_i \frac{1}{2} m_i v_i^2 = T$$

Where, T represents the total kinetic energy of the system. Now, Equation (1.28) can be written as,

$$\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} - Q_j\right] \delta q_j = 0$$

Since, the constraints are holonomic, q_j are independent of each other and hence to satisfy above equation the coefficients of each δq_j should separately vanish, i.e.

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j \qquad \ldots (1.29)$$

The equations are valid in the case of conservative as well as non-conservative forces.*These equations are called Lagrange's equations.*

For a conservative system, forces $\vec{F}_i$ are derivable from potential function V.

$$\therefore \quad \vec{F}_i = -\vec{\nabla}_i V = -\frac{\partial V}{\partial \vec{r}_i} \qquad \ldots (1.30)$$

Then generalised force can be expressed as,

$$Q_j = \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = -\sum_i \vec{\nabla}_i V \cdot \frac{\partial \vec{r}_i}{\partial q_j}$$

$$= -\sum_i \frac{\partial V}{\partial \vec{r}_i} \cdot \frac{\partial \vec{r}_i}{\partial q_j} = -\frac{\partial V}{\partial q_j}$$

$$\therefore \qquad Q_j = -\frac{\partial V}{\partial q_j} \qquad \ldots (1.31)$$

Hence, equation (1.29) becomes,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = -\frac{\partial V}{\partial q_j}$$

$$\therefore \ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial (T-V)}{\partial q_j} = 0$$

$$\therefore \ \frac{d}{dt}\left(\frac{\partial (T-V)}{\partial \dot{q}_j}\right) - \frac{\partial (T-V)}{\partial q_j} = 0$$

Since V is not a function of $\dot{q}_j$. Taking$T - V = L$, where L , is called the Lagrangian.Hence for the conservative system,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0 \qquad \dots (1.32)$$

Equation (1.32) is known as Lagrange's equations of motion for conservative system.

Importance of Lagrangian Formulation:

The derivation of Lagrange's equations for a system is equivalent to Newton's equation of motion. The Lagrangian formulation of mechanics is only an alternative and equivalent formulation. The Newtonian and Lagrangian equations of motion are the second order differential equation of motion which describe the nature of motion.

In the Newtonian approach, we are concern with the applied forces acting on a system. This force accelerates the system. The forces are due to external agencies, which produced the acceleration in the body. The resulting accelerated motion is the effect of the force.

In Lagrangian approach, we consider the kinetic and potential energies of the system. The concept of the forces does not involve directly. This is the difference between the two formulations. The kinetic and potential energies are scalar functions and invariant under coordinate transformations. These transformations include the transformation from the Cartesian coordinates $\vec{r}_i$ to the generalised coordinates q_j. In such transformation, it is easier to deal with scalar quantities than vector such as force, momentum, torque etc., which are involved in the Newtonian formulation.

Another difference is that in some problem it may not be possible to know all the forces acting on the system. But the expressions for the kinetic and potential energy may be given.

Hence, the Lagrangian formulation is more useful.

A General Expression For Kinetic Energy:

The kinetic energy of the system is given by

$$T = \sum_i \frac{1}{2} m_i \dot{r}_i^2 = \sum_i \frac{1}{2} m_i v_i^2 \qquad \dots (1.33)$$

Now, we have

$$\vec{r}_i = \vec{r}_i(q_j, t)$$

$$\therefore \quad \dot{\vec{r}}_i = \sum_j \frac{\partial \vec{r}_i}{\partial q_j} \dot{q}_j + \frac{\partial \vec{r}_i}{\partial t} \qquad \dots (1.34)$$

Squaring above equation, we get

$$\dot{r}_i^2 = \sum_{jk} \frac{\partial \vec{r}_i}{\partial q_j} \cdot \frac{\partial \vec{r}_i}{\partial q_k} \dot{q}_j \dot{q}_k + 2\sum_{j} \frac{\partial \vec{r}_i}{\partial q_j} \cdot \frac{\partial \vec{r}_i}{\partial t} \dot{q}_j + \left(\frac{\partial \vec{r}_i}{\partial t}\right)^2 \qquad \ldots (1.35)$$

Using equation (1.35) in (1.33), we get

$$T = \sum_{jk} \left(\frac{1}{2}\sum_i m_i \frac{\partial \vec{r}_i}{\partial q_j} \cdot \frac{\partial \vec{r}_i}{\partial q_k}\right) \dot{q}_j \dot{q}_k + \sum_j \left(\sum_i m_i \frac{\partial \vec{r}_i}{\partial q_j} \cdot \frac{\partial \vec{r}_i}{\partial t}\right) \dot{q}_j + \frac{1}{2}\sum_i m_i \left(\frac{\partial \vec{r}_i}{\partial t}\right)^2$$

$$\therefore T = \sum_{jk} a_{jk} \dot{q}_j \dot{q}_k + \sum_j b_j \dot{q}_j + c \qquad \ldots (1.36)$$

Where,

$$\left.\begin{aligned} a_{jk} &= \frac{1}{2}\sum_i m_i \frac{\partial \vec{r}_i}{\partial q_j} \cdot \frac{\partial \vec{r}_i}{\partial q_k} \\ b_j &= \sum_i m_i \frac{\partial \vec{r}_i}{\partial q_j} \cdot \frac{\partial \vec{r}_i}{\partial t} \\ c &= \frac{1}{2}\sum_i m_i \left(\frac{\partial \vec{r}_i}{\partial t}\right)^2 \end{aligned}\right\} \qquad \ldots (1.37)$$

When transformation equations are independent of time, then $b_j = 0$, and $c = 0$

$$\therefore T = \sum_{jk} a_{jk} \dot{q}_j \dot{q}_k \qquad \ldots (1.38)$$

Illustration:Consider a double pendulum as shown in fig.1.7

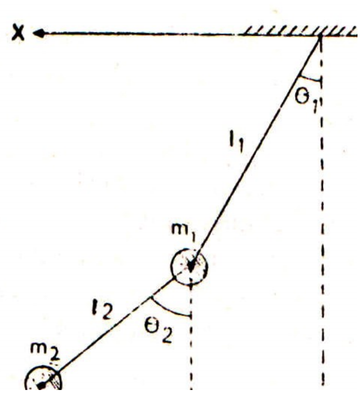

Fig. 1.7 – Double pendulum

Let l_1 and l_2 be the length of first and second pendulum respectively. Let us assume that the pendulums move in the XY plane only. Let $\theta_1$and $\theta_2$be the angular displacements of first and second pendulum respectively. The generalised coordinates are $q_1 = \theta_1$ and $q_2 = \theta_2$.

Let (x_1, y_1) and (x_2, y_2) be the coordinates of two bobs. Hence from fig. we have,

$$\left.\begin{aligned} x_1 &= l_1 sin\theta_1 \\ y_1 &= l_1 cos\theta_1 \\ x_2 &= l_1 sin\theta_1 + l_2 sin\theta_2 \\ y_2 &= l_1 cos\theta_1 + l_2 cos\theta_2 \end{aligned}\right\} \quad \text{... (1.39)}$$

Differentiation of equation (1.39) with respect to time, we have

$$\left.\begin{aligned} \dot{x}_1 &= l_1 sin\dot{\theta}_1 \\ \dot{y}_1 &= -l_1 cos\dot{\theta}_1 \\ \dot{x}_2 &= l_1 cos\theta_1\dot{\theta}_1 + l_2 cos\theta_2\dot{\theta}_2 \\ \dot{y}_2 &= -l_1 sin\theta_1\dot{\theta}_1 - l_2 sin\theta_2\dot{\theta}_2 \end{aligned}\right\} \quad \text{... (1.40)}$$

Now, kinetic energy T is given by

$$T = \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2) \quad \text{... (1.41)}$$

Substituting the values of equation (1.40) in equation (1.41), we have

$$\begin{aligned} T = \frac{1}{2}m_1\big(l_1^2 cos^2\theta_1\dot{\theta}_1^2 + l_1^2 sin^2\theta_1\dot{\theta}_1^2\big) \\ + \frac{1}{2}m_2\big(l_1^2 cos^2\theta_1\dot{\theta}_1^2 + l_2^2 cos^2\theta_2\dot{\theta}_2^2 + 2l_1 l_2 cos\theta_1 cos\theta_2\dot{\theta}_1\dot{\theta}_2 + l_1^2 sin^2\theta_1\dot{\theta}_1^2 \\ + l_2^2 sin^2\theta_2\dot{\theta}_2^2 + 2l_1 l_2 sin\theta_1 sin\theta_2\dot{\theta}_1\dot{\theta}_2\big) \end{aligned}$$

$$\therefore\ T = \frac{1}{2}(m_1 + m_2)l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2 l_2^2\dot{\theta}_2^2 + m_2 l_1 l_2 cos(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2 \quad \text{... (1.42)}$$

This expression can also be obtained by using equation (1.40)

$$\begin{aligned} T &= \sum_{jk} a_{jk}\dot{q}_j\dot{q}_k \\ &= a_{\theta_1\theta_1}\dot{\theta}_1^2 + \big(a_{\theta_1\theta_2} + a_{\theta_2\theta_1}\big)\dot{\theta}_1\dot{\theta}_2 + a_{\theta_2\theta_2}\dot{\theta}_2^2 \end{aligned} \quad \text{... (1.43)}$$

Where,

$$a_{\theta_1\theta_1} = \frac{1}{2}m_1\left[\left(\frac{\partial x_1}{\partial\theta_1}\right)^2 + \left(\frac{\partial y_1}{\partial\theta_1}\right)^2\right] + \frac{1}{2}m_1\left[\left(\frac{\partial x_2}{\partial\theta_1}\right)^2 + \left(\frac{\partial y_2}{\partial\theta_1}\right)^2\right]$$

$$a_{\theta_1\theta_2} = a_{\theta_2\theta_1} = \frac{1}{2}m_1\left[\frac{\partial x_1}{\partial\theta_1}\frac{\partial x_1}{\partial\theta_2} + \frac{\partial y_1}{\partial\theta_1}\frac{\partial y_1}{\partial\theta_2}\right] + \frac{1}{2}m_2\left[\frac{\partial x_2}{\partial\theta_1}\frac{\partial x_2}{\partial\theta_2} + \frac{\partial y_2}{\partial\theta_1}\frac{\partial y_2}{\partial\theta_2}\right]$$

$$a_{\theta_2\theta_2} = \frac{1}{2}m_1\left[\left(\frac{\partial x_1}{\partial\theta_2}\right)^2 + \left(\frac{\partial y_1}{\partial\theta_2}\right)^2\right] + \frac{1}{2}m_2\left[\left(\frac{\partial x_2}{\partial\theta_2}\right)^2 + \left(\frac{\partial y_2}{\partial\theta_2}\right)^2\right]$$

Substituting the values of derivatives, we get

$$\left.\begin{aligned} a_{\theta_1\theta_1} &= \frac{1}{2}(m_1+m_2)l_1^2 \\ a_{\theta_1\theta_2} = a_{\theta_2\theta_1} &= \frac{1}{2}m_2 l_1 l_2 cos(\theta_1-\theta_2) \\ a_{\theta_2\theta_2} &= \frac{1}{2}m_1 l_2^2 \end{aligned}\right\} \qquad \dots(1.44)$$

With this substitution in equation (1.43), we get same expression of K.E as

$$T = \frac{1}{2}m_1(\dot{x}_1^2+\dot{y}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2+\dot{y}_2^2) \qquad \dots(1.45)$$

Symmetries and the Laws of Conservation:

The study of conservation theorems for a system in motion provides the constants of motion. The knowledge of which helps in describing the motion of the system. The Lagrange's equations of motion are very appropriate for recognizing the constants of motion.

Lagrange's equations of motion are second order in time. To solve them completely, for a system of n degrees of freedom, $2n$ constants of integrations will be involved which can be determined if n initial values for q_j and n initial values for $\dot{q}_j$ are known. It is not always possible to integrate every equation of motion so as to get complete solution in terms of known functions. But in such case it is possible to extract sufficient information about the physical nature of the motion of the system without solving the problem completely through the constants of motion.

On considering the symmetries of the system, we can obtain first integrals of the equations of motion. The first integrals are constants of motion. These are the first order differential equation of the type

$$f(q_1, q_2, \dots\dots q_n, \dot{q}_1, \dot{q}_2, \dots\dots \dot{q}_n, t) = constant \qquad \dots(1.46)$$

The first integrals provide a lot of information regarding the system.

Consider a system of particles in a conservative force field. Then, potential energy V depends only upon the position.

$$\therefore \ \frac{\partial L}{\partial \dot{x}_i} = \frac{\partial (T-V)}{\partial \dot{x}_i} = \frac{\partial}{\partial \dot{x}_i}\sum_i \frac{1}{2}m_i(\dot{x}_i^2+\dot{y}_i^2+\dot{z}_i^2)$$

$$\therefore \ \frac{\partial L}{\partial \dot{x}_i} = m_i\dot{x}_i = p_{x_i}$$

Where, p_{x_i} is the x- component of the linear momentum of the i^{th} particle.

We can generalize this result and define the generalized momentum by the formula

$$p_j = \frac{\partial L}{\partial \dot{q}_j} \quad \dots (1.47)$$

Where, p_j is called the *canonical* or *conjugate momentum*.

If q_j is linear displacement the p_j represents the linear momentum. But, if q_j represent an angle then p_j represents the angular momentum.

If we consider a charged particle moving in an electromagnetic field, then the Lagrangian is defined as

$$L = \frac{1}{2} m \left|\dot{\vec{r}}\right|^2 - q\emptyset + q\vec{A} \cdot \dot{\vec{r}} \quad \dots (1.48)$$

Where, $\emptyset$ is scalar potential and $\vec{A}$ is a vector potential. Differentiate equation (1.48) with respect to $\dot{x}$, we get

$$p_x = m\dot{x} + qA_x \quad \dots (1.49)$$

Ii is not the useful kinetic momentum $m\dot{x}$ but has a contribution of qA_x from the electromagnetic field.

Lagrange's equations for a conservative system assume a simple form similar to Newton's equations of motion as follows

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$$

$$\therefore \quad \frac{d}{dt}(p_j) = \frac{\partial L}{\partial q_j}$$

$$\therefore \; \dot{p}_j = \frac{\partial (T - V)}{\partial q_j} = \frac{\partial T}{\partial q_j} - \frac{\partial V}{\partial q_j} \quad \dots (1.50)$$

For conservative system, the potential energy depends only on position coordinate q_j. Hence, $\frac{\partial T}{\partial q_j} = 0$. Therefore, above equation (1.50) becomes

$$\dot{p}_j = -\frac{\partial V}{\partial q_j}$$

$$\therefore \quad \dot{p}_j = Q_j \quad \dots (1.51)$$

Cyclic or Ignorable Coordinates:

The Lagrangian$L$ is a function of q_j and $\dot{q}_j$. If anyone coordinate, say q_k is absent in the expression of Lagrangian L , then

$$\frac{\partial L}{\partial q_k} = 0$$

The equation of motion corresponding to variable q_k becomes

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = 0$$

Integrating above equation, we get

$$\frac{\partial L}{\partial \dot{q}_k} = p_k = constant \qquad \dots (1.52)$$

Thus, when coordinate q_k does not appear in the Lagrangian function L , then corresponding linear momentump_k is a constant of the motion. Such coordinate q_k is said to be cyclic or ignorable coordinate.

For conservation of energy following two conditions must be satisfied.

(i) The potential energy must be a function of coordinate only and not that of a velocities,

(ii) The constraints do not change with time, i.e. they are independent of time and the equations of transformation to generalised coordinates do not involve time explicitly.

$$\therefore \quad L = L\left(q_j, \dot{q}_j\right)$$

Its total time derivative will be

$$\frac{dL}{dt} = \sum_j \frac{\partial L}{\partial q_j}\frac{dq_j}{dt} + \sum_j \frac{\partial L}{\partial \dot{q}_j}\frac{d\dot{q}_j}{dt} \qquad \dots (1.53)$$

But, Lagrange's equation of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$$

$$\therefore \; \frac{\partial L}{\partial q_j} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) \qquad \dots (1.54)$$

Putting this value of $\frac{\partial L}{\partial q_j}$ from equation (1.54) in equation (1.53), we get

$$\frac{dL}{dt} = \sum_j \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\dot{q}_j + \frac{\partial L}{\partial \dot{q}_j}\frac{d\dot{q}_j}{dt}\right]$$

$$\therefore \; \frac{dL}{dt} = \sum_j \frac{d}{dt}\left(\dot{q}_j \frac{\partial L}{\partial \dot{q}_j}\right) = \sum_j \frac{d}{dt}\left(\dot{q}_j \frac{\partial (T-V)}{\partial \dot{q}_j}\right)$$

$$\therefore \quad \frac{dL}{dt} = \sum_j \frac{d}{dt}\left(\dot{q}_j \frac{\partial T}{\partial \dot{q}_j} - \dot{q}_j \frac{\partial V}{\partial \dot{q}_j}\right) \qquad \ldots (1.55)$$

But, for conservative system, $\frac{\partial V}{\partial \dot{q}_j} = 0$. Hence above equation (1.55) reduce to

$$\frac{dL}{dt} - \sum_j \frac{d}{dt}(\dot{q}_j p_j) = 0$$

$$\therefore \quad \frac{d}{dt}\left[\sum_j \dot{q}_j p_j - L\right] = 0$$

$$\therefore \quad \sum_j \dot{q}_j p_j - L = constant = H \qquad \ldots (1.56)$$

The quantity H is one of the first integrals of equation of motion and it represents the total energy of the system.

When constraints are independent of time i.e. equation of transformation do not involve time explicitly and constraints are holonomic, then kinetic energy can be expressed as homogeneous quadratic function of generalised velocities and therefore,

$$T = \sum_{jk} a_{jk} \dot{q}_j \dot{q}_k$$

Euler's theorem states that if f is a homogeneous function of order n of a set of variableq_j , then

$$\sum_j q_j \frac{\partial f}{\partial q_j} = nf$$

But, here $n = 2$,so that

$$\sum_j \dot{q}_j \frac{\partial T}{\partial \dot{q}_j} = 2T$$

$$\therefore \quad \sum_j \dot{q}_j p_j = 2T \qquad \ldots (1.57)$$

Thus, using above equation (1.57) in equation (1.56), we get

$$H = 2T - L = 2T - (T - V)$$

$$\therefore \quad H = T + V = E \qquad \ldots (1.58)$$

Which shows that H equals the total energy and is conserved.

It may happen that H be a constant of motion but not the total energy.

Suppose the transformation equation involve time, then

$$\sum_j \dot{q}_j \frac{\partial T}{\partial \dot{q}_j} \neq 2T$$

$$\therefore \quad H \neq T + V$$

But, $\frac{d}{dt}\left[\sum_j \dot{q}_j p_j - L\right] = 0$

$$\therefore \quad \frac{dH}{dt} = 0$$

$$\therefore \quad H = Constant \qquad \ldots(1.59)$$

$\therefore H$ is still conserved i.e. it continues to be a constant of motion. Therefore, identification of H as a constant of motion and as the total energy are two separate matters.

ILLUSTRATIONS:

(1) Motion of a free particle:

Consider the motion of a particle due to force $\vec{F}$ having componentsF_x, F_y and F_z along the three axes of the Cartesian coordinate.

Hence, the kinetic energy is given by

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \qquad \ldots(1.60)$$

And, the potential energy for free particle is,

$$V = 0$$

In this case, the Lagrangian$L = T - V = T$

The equation of motion can be written as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = 0 \qquad \ldots(1.61)$$

Here, $j = 1,2,3$. Let us consider the generalised coordinate $q_1 = x, q_2 = y\ and\ q_3 = z$

The Lagrange's equations of motions are

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} = 0, \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}}\right) - \frac{\partial T}{\partial y} = 0, \quad and\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{z}}\right) - \frac{\partial T}{\partial z} = 0$$

In this case,

$$\frac{\partial T}{\partial x} = \frac{\partial T}{\partial y} = \frac{\partial T}{\partial z} = 0 \; and$$

$$\frac{\partial T}{\partial \dot{x}} = m\dot{x}, \quad \frac{\partial T}{\partial \dot{y}} = m\dot{y} \quad and \quad \frac{\partial T}{\partial \dot{z}} = m\dot{z}$$

Hence, the equations of motions are,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) = \frac{d}{dt}(m\dot{x}) = m\ddot{x} = F_x$$

Similarly,

$$m\ddot{y} = F_y \; and \; m\ddot{z} = F_z$$

These are Newton's equations of motion.

(2) Motion of bead along rotating wire:

Consider a bead sliding along a uniformly rotating wire in a force-free space. The transformation equations relating the Cartesian and polar coordinates of the bead are

$$x = r\cos\theta = r\cos\omega t$$

$$and, \qquad y = r\sin\theta = r\sin\omega t$$

Where, ω is the constant angular velocity of the wire.

Kinetic energy T is given by

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

$$= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$$

$$\therefore T = \frac{1}{2}m(\dot{r}^2 + r^2\omega^2) \qquad \ldots (1.62)$$

In this case, the Lagrangian $L = T - V = T$

Because the potential energy $V = 0$

Thus, we find that T is not a homogeneous quadratic function of velocities only. The equation of motion can be written as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$$

Here, choosing the generalised coordinate $q_j = r$, and $L = T$

Above equation becomes,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}}\right) - \frac{\partial T}{\partial r} = 0$$

$$\therefore \quad m\ddot{r} - mr\omega^2 = 0$$

$$\therefore \quad \ddot{r} = r\omega^2 \qquad \dots (1.63)$$

This is the familiar expression of the centripetal acceleration.

(3) Atwood's machine:

Atwood's machine consists of two masses m_1 and m_2 tied together by a light cord oflength l as shown in figure 1.8 The cord passes round a light frictionless pulley and the two masses hang on the two sides of the pulley.

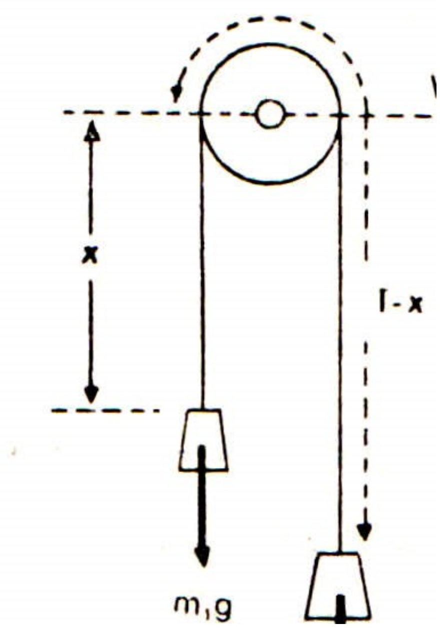

Fig. 1.8 – Atwood's machine

Here, only one variable x is independent and l is constant. The kinetic and potential energies of the system are given by

$$T = \frac{1}{2}(m_1 + m_2)\dot{x}^2$$

and, $V = -m_1 g x - m_2 g(l - x)$

The Lagrangian$L = T - V$

$$\therefore L = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + m_1 g x + m_2 g(l - x) \qquad \dots (1.64)$$

The Lagrange's equation of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)-\frac{\partial L}{\partial q_j}=0$$

Here, choosing the generalised coordinate $q_j = x$

$$\therefore \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)-\frac{\partial L}{\partial x}=0$$

$$\therefore \frac{d}{dt}[(m_1+m_2)\dot{x}]-m_1x-m_2g(-1)=0$$

$$\therefore (m_1+m_2)\ddot{x}-(m_1-m_2)g=0$$

$$\therefore \ddot{x}=\frac{m_1-m_2}{m_1+m_2}g \qquad \ldots(1.65)$$

This is an acceleration of Atwood's machine.

Integrating above equation twice, we get

$$x=x_0+v_0t+\frac{1}{2}\left(\frac{m_1-m_2}{m_1+m_2}\right)gt^2 \qquad \ldots(1.66)$$

This is the displacement of Atwood's machine.

(4) Spherical Pendulum:

A spherical pendulum is a simple pendulum which is free to move through the entirespace about the point of suspension as shown in figure 1.9

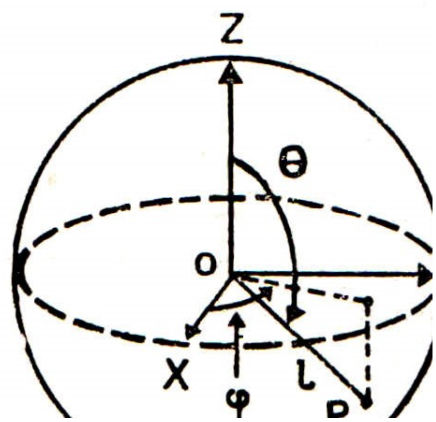

Fig. 1.9 – Spherical pendulum

The bob of a pendulum moves on the surface of a sphere whose radius is equal to the length of the pendulum. The spherical polar coordinates are suitable to locate the position of the bob. Here, l is constant and $\theta, \emptyset$ are variables.

The velocity of the bob is given by

$$\vec{v}=l\dot{\theta}\hat{e}_\theta+l\,sin\theta\dot{\emptyset}\hat{e}_\emptyset$$

The kinetic energy becomes$T=\frac{1}{2}mv^2$

$$\therefore T = \frac{1}{2}m\left(l^2\dot{\theta}^2 + l^2 sin^2\theta\dot{\emptyset}^2\right)$$

The potential energy $V = -mgl\, cos\theta$

The Lagrangian $L = T - V$

$$\therefore L = \frac{1}{2}ml^2\dot{\theta}^2 + \frac{1}{2}ml^2 sin^2\theta\dot{\emptyset}^2 + mgl\, cos\theta \qquad \dots (1.67)$$

The generalised coordinates are θ and $\emptyset$

Therefore, the Lagrange's equation of motion corresponding to θ and $\emptyset$ are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0 \qquad \dots (1.68)$$

And

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\emptyset}}\right) - \frac{\partial L}{\partial\emptyset} = 0 \qquad \dots (1.69)$$

The solution of equation (1.68) is

$$\frac{d}{dt}\left[ml^2\dot{\theta}\right] - ml^2 sin\theta\, cos\theta\, \dot{\emptyset}^2 - mgl\, sin\theta = 0$$

$$\therefore ml^2\ddot{\theta} = ml^2 sin\theta\, cos\theta\dot{\emptyset}^2 + mgl\, sin\theta \qquad \dots (1.70)$$

The solution of equation (1.69) is

$$\frac{d}{dt}\left(ml^2 sin^2\theta\, \dot{\emptyset}\right) = 0$$

$$\therefore ml^2 sin^2\theta\, \ddot{\emptyset} = 0 \qquad \dots (1.71)$$

Equations (1.70) and (1.71) are the equation of motion of spherical pendulum. Variable $\emptyset$ is ignorable, hence its corresponding momentum is conserved.

$$\therefore\ p_{\emptyset} = \frac{\partial L}{\partial\dot{\emptyset}} = ml^2 sin^2\theta\dot{\emptyset} = constant \qquad \dots (1.72)$$

Substituting the value of $\dot{\emptyset}$ from equation (1.72) in equation (1.71)

$$\therefore\ ml^2\, \ddot{\emptyset} = mgl\, sin\theta + ml^2 sin\theta\, cos\theta\, \frac{{p_{\emptyset}}^2}{m^2 l^4 sin^4\theta}$$

$$\therefore\ ml^2\, \ddot{\emptyset} = mgl\, sin\theta + \frac{{p_{\emptyset}}^2 cos\theta}{ml^2 sin^3\theta} \qquad \dots (1.73)$$

Since, $\frac{\partial L}{\partial t} = 0$

The total energy is given by

$$E = \sum_i \dot{q}_i p_i - L \qquad \text{... (1.74)}$$

The generalised coordinates are θ and $\emptyset$. Hence, above equation becomes,

$$E = \dot{\theta} p_\theta + \dot{\emptyset} p_\emptyset - L$$

$$\therefore \quad E = \dot{\theta}\frac{\partial L}{\partial \dot{\theta}} + \dot{\emptyset}\frac{\partial L}{\partial \dot{\emptyset}} - L$$

$$\therefore E = \dot{\theta} ml^2\dot{\theta} + \dot{\emptyset}\, ml^2 sin^2\theta\dot{\emptyset} - \frac{1}{2}ml^2\dot{\theta}^2 - \frac{1}{2}ml^2 sin^2\theta\dot{\emptyset}^2 - mgl\, cos\theta$$

$$\therefore E = \frac{1}{2}ml^2\dot{\theta}^2 + \frac{1}{2}ml^2 sin^2\theta\dot{\emptyset}^2 + mgl\, cos\theta$$

$$\therefore E = T + V = constant \qquad \text{... (1.75)}$$

It is the constant of motion.

The expression of total energy can be rewrite as,

$$E = \frac{1}{2}ml^2\dot{\theta}^2 + \frac{1}{2}ml^2 sin^2\theta\ \frac{p_\emptyset^2}{m^2 l^4 sin^4\theta} + mgl\, cos\theta$$

$$\therefore E = \frac{1}{2}ml^2\dot{\theta}^2 + \frac{p_\emptyset^2}{2ml^2 sin^2\theta} + mgl\, cos\theta$$

$$\therefore E = T + V_e \qquad \text{... (1.76)}$$

Where, $V_e = \frac{p_\emptyset^2}{2ml^2 sin^2\theta} + mgl\, cos\theta$ is the effective potential energy and it is depends only on θ.

- If the pendulum is restricted to move only in one plane, then $\emptyset = 0$. Theequation of motion (1.70) reduce to

$$ml^2\ddot{\theta} = mgl\, sin\theta$$

$$\therefore \quad \ddot{\theta} - \left(\frac{g}{l}\right) sin\theta = 0 \qquad \text{... (1.77)}$$

This is equation of motion of simple pendulum.

Velocity Dependent Potential of Electromagnetic Field:

The Lagrange's equation of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0 \qquad \text{... (1.78)}$$

If the system is not conservative then the generalize forces are expressed as

$$Q_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right) \qquad \dots (1.79)$$

Where, $U(q_j, \dot{q}_j)$ is called *generalised potential* or *velocity dependent potential*.

This type potential involve in case of the electromagnetic forces acting on moving charges.

The Maxwell's equations are,

$$\left.\begin{aligned} \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \\ \vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \\ \vec{\nabla} \times \vec{B} = \mu_0 j + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \\ \vec{\nabla} \cdot \vec{B} = 0 \end{aligned}\right\} \qquad \dots (1.80)$$

Here, $\vec{B} = \mu_0 H$ is *magnetic induction* and the quantity $\epsilon_0 \vec{E} = \vec{D}$ is called the *electric displacement*.

The force on a charged particle having charge q and moving in an electromagnetic field is given by

$$\vec{F} = q[\vec{E} + \vec{v} \times \vec{B}] \qquad \dots (1.81)$$

This force is known as the Lorentz force.

We know the properties if gradient, divergence and curl. In which we know that,

$$\vec{\nabla} \cdot \vec{\nabla} \times \vec{A} = 0 \qquad \dots (1.82)$$

But, one of the Maxwell's equations is

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \dots (1.83)$$

Comparing equations (1.82) and (1.83), we get

$$\vec{B} = \vec{\nabla} \times \vec{A} \qquad \dots (1.84)$$

Where $\vec{A}$ is called a magnetic vector potential. With this substitution for $\vec{B}$ in the following equation, we have

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

$$\therefore \ \vec{\nabla} \times \vec{E} + \frac{\partial}{\partial t}(\vec{\nabla} \times \vec{A}) = 0$$

$$\therefore\ \vec{\nabla} \times \left[\vec{E} + \frac{\partial \vec{A}}{\partial t}\right] = 0 \qquad \dots (1.85)$$

We have another property of gradient and curl is,

$$\vec{\nabla} \times \vec{\nabla}\emptyset = 0 \qquad \dots (1.86)$$

Comparing equation (1.85) and (1.86) we get,

$$\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla}\emptyset$$

Here, $(-)$ sign indicates that potential decreases away from charged particle.

$$\therefore\ \vec{E} = -\vec{\nabla}\emptyset - \frac{\partial \vec{A}}{\partial t} \qquad \dots (1.87)$$

Now, the Lorentz force can be written as,

$$\vec{F} = q\left[-\vec{\nabla}\emptyset - \frac{\partial \vec{A}}{\partial t} + \vec{v} \times \vec{\nabla} \times \vec{A}\right] \qquad \dots (1.88)$$

Now, substituting the values of $\vec{\nabla}\emptyset$ and $\vec{v} \times \vec{\nabla} \times \vec{A}$, the x - component of the Lorentz force is given by,

$$F_x = q\left[-\frac{\partial}{\partial x}\{\emptyset - \vec{v} \cdot \vec{A}\} - \frac{d}{dt}\left\{\frac{\partial}{\partial v_x}(\vec{A} \cdot \vec{v})\right\}\right] \qquad \dots (1.89)$$

The scalar potential $\emptyset$ is independent of velocity. Hence above equation can be written as,

$$F_x = q\left[-\frac{\partial}{\partial x}\{\emptyset - \vec{A} \cdot \vec{v}\} - \frac{d}{dt}\left\{\frac{\partial}{\partial v_x}\{\emptyset - \vec{A} \cdot \vec{v}\}\right\}\right]$$

$$\therefore\ F_x = -\frac{\partial}{\partial x}\{q(\emptyset - \vec{A} \cdot \vec{v})\} + \frac{d}{dt}\frac{\partial}{\partial v_x}\{q(\emptyset - \vec{A} \cdot \vec{v})\}$$

$$\therefore\ F_x = -\frac{\partial U}{\partial x} + \frac{d}{dt}\frac{\partial U}{\partial v_x} \qquad \dots (1.90)$$

Here,

$$U = q\emptyset - q\vec{A} \cdot \vec{v} \qquad \dots (1.91)$$

Where, U is the generalised potential energy.

Hence, the Lagrangian for a charged particle moving in an electromagnetic field is given by

$$L = T - U = T - q\emptyset + q\vec{A}\cdot\vec{v} \qquad \dots (1.92)$$

Rayleigh's Dissipation Function:

If the force acting on the systems is derivable from a potential, then Lagrange's equation becomes

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} \qquad \dots (1.93)$$

Where L contains only those forces that are conservative while Q_j includes the forces that are not derivable from a potential like frictional force.

When the frictional force is proportional to the velocity, then $x -$ component of the force is given by

$$F_{fx} = -k_x v_x \qquad \dots (1.94)$$

Where, k_x is the $x -$ component of the frictional force per unit velocity in x-direction.

This type of forces are derives from Rayleigh's dissipation function $\mathcal{F}$ defined by

$$\mathcal{F} = \frac{1}{2}\sum_i \left(k_x v_{ix}^2 + k_y v_{iy}^2 + k_z v_{iz}^2\right) \qquad \dots (1.95)$$

Where, $i = 1,2, \dots \dots \dots n$

Now,
$$F_f = -\vec{\nabla}_v \mathcal{F} \qquad \dots (1.96)$$

Where,

$$\vec{\nabla}_v = \hat{\imath}\frac{\partial}{\partial v_x} + \hat{\jmath}\frac{\partial}{\partial v_y} + \hat{k}\frac{\partial}{\partial v_z} \qquad \dots (1.97)$$

This is the vector velocity differential operator.

To explain the physical significance of Rayleigh's dissipation function. Let us calculate the work done by the system against friction as

$$dW_f = -\vec{F}_f \cdot d\vec{r} = -\vec{F}_f \cdot \vec{v}dt$$

$$\therefore\ dW_f = -\left(\hat{\imath}\, k_x v_x + \hat{\jmath} k_y v_y + \hat{k} k_z v_z\right)\cdot\left(\hat{\imath}\, v_x + \hat{\jmath} v_y + \hat{k} v_z\right)dt$$

$$\therefore\ dW_f = \left(k_x v_x^2 + k_y v_y^2 + k_z v_z^2\right)dt$$

$$\therefore\ \frac{dW_f}{dt} = k_x v_x^2 + k_y v_y^2 + k_z v_z^2$$

$$\therefore \frac{dW_f}{dt} = 2\mathcal{F} \qquad \dots (1.98)$$

Thus, the rate of dissipation of energy by friction is equal to twice Rayleigh's dissipation function.

The component Q_j of the generalised force arise due to frictional force and is given by

$$Q_j = \sum_i F_{fi} \cdot \frac{\partial \vec{r}_i}{\partial q_j}$$

$$\therefore Q_j = -\sum \vec{\nabla}_v \mathcal{F} \cdot \frac{\partial \vec{r}_i}{\partial q_j} = -\sum \vec{\nabla}_v \mathcal{F} \cdot \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j}$$

$$\therefore Q_j = -\frac{\partial \mathcal{F}}{\partial \dot{q}_j} \qquad \dots (1.99)$$

Using equation (1.99) in equation (1.93), the Lagrange's equation of motion is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} + \frac{\partial \mathcal{F}}{\partial \dot{q}_j} = 0 \qquad \dots (1.100)$$

Thus, if frictional force is acting on the system, we must specify two scalar functions, the Lagrangian$L$ and Rayleigh's dissipation function $\mathcal{F}$ to derive the equation of motion.

Question Bank

Multiple choice questions:

(1) The degree of freedom for a free particle in space are _________
(a) one (b)two
(c)**three** (d) zero

(2) The number of independent variable for a free particle in space are ______
(a) zero (b) one
(c)two (d) **three**

(3) The degree of freedom for N particles in space are _________
(a) 2N (b)**3N**
(c)N (d) zero

(4) The number of independent variable for a free particle in space are ______
(a) N (b) 2N
(c)**3N** (d) zero

(5) ___________ constraints are independent of time.
(a) Holonomic (b) Non-Holonomic
(c)**Scleronomous** (d) Rheonomous

(6) ___________ constraints are time dependent.
(a) Holonomic (b) Non-Holonomic
(c)Scleronomous (d) **Rheonomous**

(7) The generalized coordinates for motion of a particle moving on the surface of a sphere of radius 'α ' are _________
(a) αand θ (b)αand ϕ
(c)**θ and ϕ** (d)0and ϕ

(8) The Lagrangian equations of motion are ___________ order differential equations.
(a) first (b) **second**
(c) zero (d) forth

(9) The Lagrange's equations of motion for a system is equivalent to _______ equations of motion.
(a) **Newton's** (b) Laplace
(c) Poisson (d) Maxwell's

(10) The Lagrangian function is define by ____________
(a) $L = F + V$ (b)$\boldsymbol{L = T - V}$
(c)$L = T + V$ (d) $L = F - V$

(11) The Hamiltonian function is define by ____________
(b) $H = F + V$ (b)$H = T - V$
(c)$\boldsymbol{H = T + V}$ (d) $H = F - V$

Short Questions:

1. Define constraint motion.
2. What is degree of freedom?
3. What is virtual displacement?
4. Define Holonomic and non-holonomic constraints.
5. Define Scleronomous and Rheonomous constraints.
6. State the D'Alembert's principle in words.
7. Write the Lagrange's equation of motion for conservative system.
8. Write the Lagrange's equation of motion for non-conservative system.
9. Define cyclic coordinates.

10. Construct the Lagrangian for Atwood's machine.
11. Construct the Lagrangian for Spherical pendulum.

Long Questions:

1. What are constraints? Explain, giving examples, the meaning of holonomic and nonholonomic constraints.
2. Explain the meaning of Scleronomous and Rheonomous constraints. Give illustrations of each.
3. Is the Lagrangian formulation more advantageous than the Newtonian formulation? Why?
4. What do you understand by cyclic coordinates? Show that the generalized momentum corresponding to a cyclic coordinate is a constant of motion.
5. Explain the term 'virtual displacement' and state the principle of virtual work.
6. Describe the use of Rayleigh's dissipation function.
7. Define the Hamiltonian. When is it equal to the total energy of the system? When is it conserved?
8. What is meant by a configuration space? How is this concept used to describe the motion of a system of particles?
9. What are constraints? Discuss holonomic and Non-holonomic constraints with illustration.
10. Discuss various types of constraints with illustration
11. Discuss the concept of generalized coordinates with illustrations.
12. Discuss the virtual work done for motion of a system and derive the mathematical statement of D'Alembert's statement.
13. Derive the Lagrange's equation of motion for a conservative system from D'Alembert's principle.
14. Derive the general expression of kinetic energy and find the kinetic energy of double pendulum from it.
15. What is cyclic coordinates? Show that total energy is conserved.
16. Construct the Lagrangian of Atwood machine and derive its the equation of motion.
17. Construct the Lagrangian of spherical pendulum and derive its the equation of motion. Also show the conservation of total energy and constant of motion.